Quantitative Convergence and Stability of Seismic Inverse Problems.

Florian Faucher, Hélène Barucq, Henri Calandra, Guy Chavent

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Florian Faucher[1],

Hélène Barucq[1], Henri Calandra[2] and Guy Chavent[1].

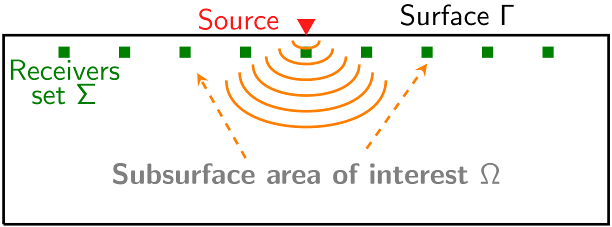
Reconstruction Methods for Inverse Problems
Indam Workshop, Roma, Italy,
May 28th – June 1st, 2018.

Istituto Nazionale di Alta Matematica

[1] Inria Bordeaux Sud-Ouest, Project-Team Magique-3D.
[2] Total E& P

Seismic inverse problem

Reconstruction of subsurface Earth properties from seismic campaign: collection of **wave** propagation data at the surface.



- Reflection (back-scattered) partial data,
- only from the surface of the (large) domain,

nonlinear, ill-posed inverse problem.

Overview

1. Time-Harmonic Inverse Problem, FWI
2. Quantitative stability and convergence of FWI
   - Finite Curvature/Limited Deflection problem
   - Numerical convergence estimates
   - Numerical stability estimates
3. Numerical experiments
4. Conclusion

Plan

1. Time-Harmonic Inverse Problem, FWI

Time-harmonic wave equation

The forward problem wave equation depends on the medium:

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- acoustic isotropic (c)

$$(-\omega^2 c^{-2} - \Delta)p = 0,$$

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- acoustic isotropic (c)
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- elastic isotropic (λ, μ, ρ)
$$-\rho\omega^2 \boldsymbol{u} - \nabla\left(\lambda \nabla \cdot \boldsymbol{u}\right) - \nabla \cdot \left(\mu\left[\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T\right]\right) = 0.$$

Time-harmonic wave equation

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- acoustic isotropic (c)

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- anisotropy (stiffness tensor, ρ)

$$-\omega^2 \rho \boldsymbol{u} - \nabla \cdot \left(\underline{\underline{C}}\,\epsilon(\boldsymbol{u})\right) = 0.$$

Viscous behavior are considered with complex coefficients.

Full Waveform Inversion (FWI)

FWI provides a quantitative reconstruction of the subsurface by solving a minimization problem,

$$\min_{m \in \mathcal{M}} \quad \mathcal{J}(m) = \frac{1}{2}\|F(m) - d\|^2.$$

- d are the observed data,
- $F(m)$ represents the simulation using an initial model m:

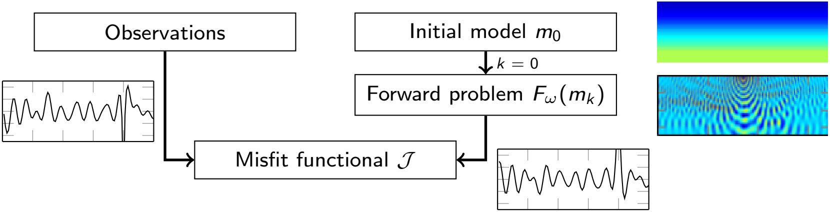
$$F \; : \; m \; \rightarrow \; \left\{p(\boldsymbol{x}_1), p(\boldsymbol{x}_2), \ldots, p(\boldsymbol{x}_{n_{rcv}})\right\}.$$

P. Lailly
The seismic inverse problem as a sequence of before stack migrations
Conference on Inverse Scattering: Theory and Application, SIAM, 1983

A. Tarantola
Inversion of seismic reflection data in the acoustic approximation
Geophysics, 1984

A. Tarantola
Inversion of travel times and seismic waveforms
Seismic tomography, 1987

FWI, iterative minimization



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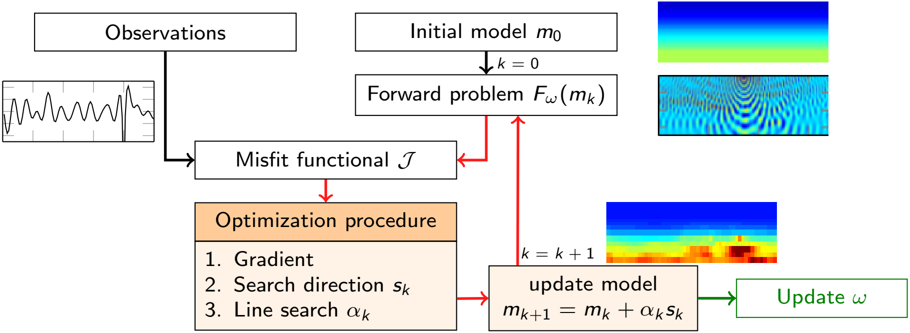

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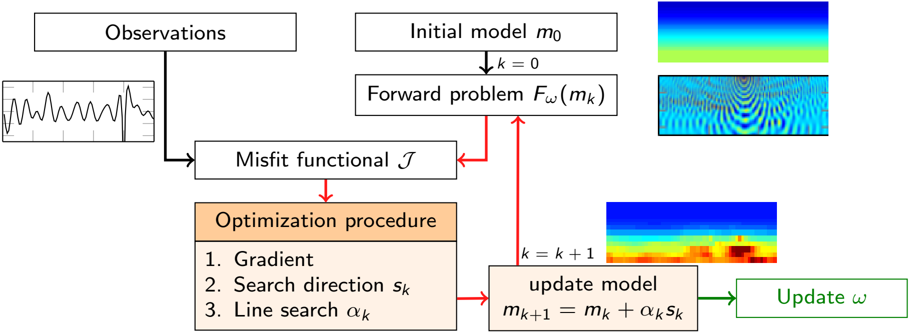

Numerical methods

- Forward problem resolution with Discontinuous Galerkin methods,
- parallel computation, HPC, large-scale optimization.

- Multi-frequency algorithm, **stability** and **convergence**.

Plan

2. Quantitative stability and convergence of FWI
   - Finite Curvature/Limited Deflection problem
   - Numerical convergence estimates
   - Numerical stability estimates

Stability & Convergence

Helmholtz inverse problem from back-scattered **partial** data,
quantitative reconstruction with iterative optimization.

$$\min_{m \in \mathcal{M}} \quad \mathcal{J}(m) = \frac{1}{2}\|\mathcal{F}(m) - d\|^2.$$

Convergence radius

- initial model needs to be within the radius of convergence.

Stability

- ensures reconstruction accuracy.

Stability & Convergence

Helmholtz inverse problem from back-scattered **partial** data, **quantitative reconstruction with iterative optimization**.

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Stability

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Numerical estimates to guide the procedure

- frequency;
- parametrization;
- geometry;
- forward problem, ...

Convergence (Finite Curvature/Limited Deflection)

1/ Convergence radius

- initial model needs to be within the radius of convergence.

Least squares minimization problem

$$\min_{m \in \mathcal{M}} \quad \mathcal{J}(m) = \frac{1}{2}\|\mathcal{F}(m) - d\|^2.$$

M. V. de Hoop, L. Qiu, O. Scherzer
An analysis of a multi-level projected steepest descent iteration for nonlinear inverse problems in Banach spaces subject to stability constraints
Numerische Mathematik 2015

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Nonlinear least squares for inverse problems: theoretical foundations and step-by-step guide for applications.
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On weakly nonlinear inverse problems.
SIAM Journal on Applied Mathematics 1996

Convergence (Finite Curvature/Limited Deflection)

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Least squares minimization problem

$$\min_{m \in \mathcal{M}} \quad \mathcal{J}(m) = \frac{1}{2} \|\mathcal{F}(m) - d\|^2.$$

Finite Curvature/Limited Deflection guarantees **uniqueness** of the solution and **unimodality**: no local minimum.

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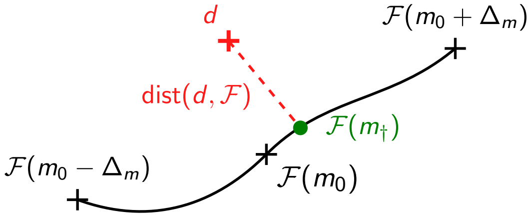
Condition 1/2: Finite Curvature problem



$$\forall m_0, \Delta_m \in \mathcal{M}, \qquad P \;:\; t \in [0,1] \to \mathcal{F}\big(m_0 + (2t-1)\Delta_m\big).$$

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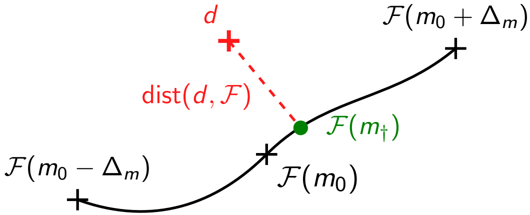
Condition 1/2: Finite Curvature problem



$$\forall m_0, \Delta_m \in \mathcal{M}, \qquad P \ : \ t \in [0,1] \to \mathcal{F}\big(m_0 + (2t-1)\Delta_m\big).$$

Finite Curvature controls the distance between data and attainable set,

$$\mathrm{dist}(d,\mathcal{F}) < \frac{\|P'(t)\|^2}{\|P''(t)\|}.$$

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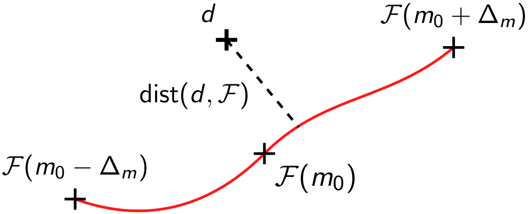
Condition 2/2: Limited Deflection problem



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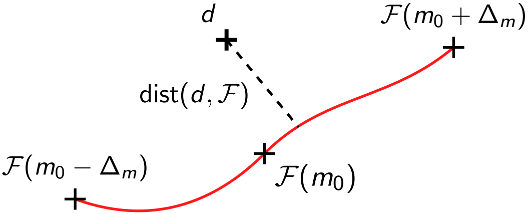

$$\forall m_0, \Delta_m \in \mathcal{M}, \qquad P \ : \ t \in [0,1] \to \mathcal{F}\big(m_0 + (2t-1)\Delta_m\big).$$

Limited Deflection property controls the attainable set and model space,

$$\Theta(P) \leq \int_0^1 \frac{\|P''(t)\|}{\|P'(t)\|}\,\mathrm{d}t. \quad \text{FC/LD requires } \boxed{\Theta(P) \leq \frac{\pi}{2}}.$$

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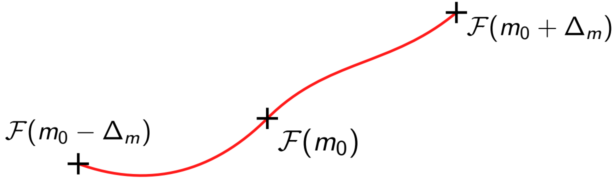
Nonlinear least squares for inverse problems: theoretical foundations and step-by-step guide for applications.
Springer 2010

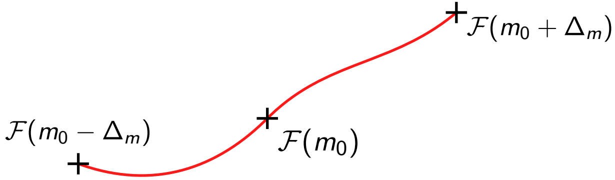
Model space size estimation via Limited Deflection



For a given m_0, we estimate the maximal distance Δ_m that still verifies the Limited Deflection property $\Theta(P) \leq \dfrac{\pi}{2}$.

Larger size are required when missing a priori info.

Model space size estimation via Limited Deflection



For a given m_0, we estimate the maximal distance Δ_m that still verifies the Limited Deflection property $\Theta(P) \leq \frac{\pi}{2}$.

- Δ_m is estimated in the direction δ_k,

$$\boxed{\|\Delta_m^{\delta_k}\| \leq \frac{\pi}{4} \frac{\|\mathcal{F}'(m_0)(\delta_k)\|}{\|\mathcal{F}''(m_0)(\delta_k, \delta_k)\|}.}$$

Larger size are required when missing a priori info.

Estimation of the basin of attraction

Numerical estimate of the size $\Delta_m^{\delta_k}$

1. with direction δ_k (the geometry of the unknown),
2. with the frequency ω,
3. with the parametrization.

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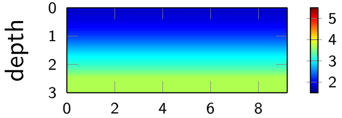
context: Helmholtz equation with back-scattered data

$$(-\omega^2 m - \Delta)p = 0,$$



m_0 is a smooth velocity

- reflection data (top free surface),
- absorbing conditions on the sides,
- piecewise constant m.

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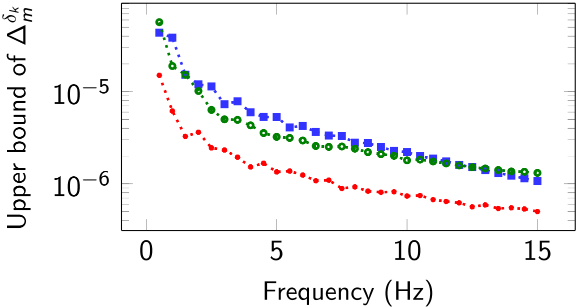
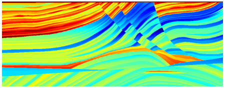
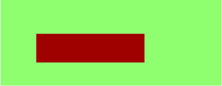


(a) direction δ_1



(b) direction δ_2



(c) direction δ_3

Estimation of the basin of attraction

Numerical estimate of the size $\Delta_m^{\delta_k}$ with

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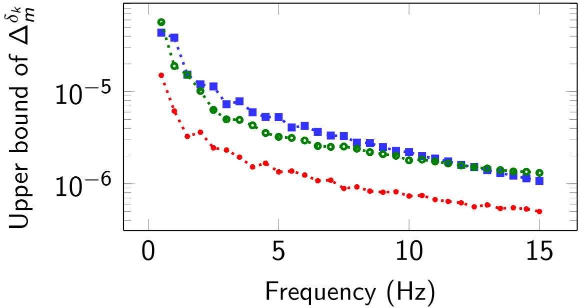

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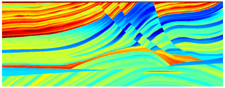
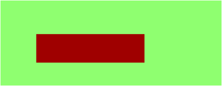


(b) direction δ_2



(c) direction δ_3

Low frequencies increase the size.

Reflecting objects complicate the procedure.

Complex frequency

$$(-\omega^2 c^{-2} - \Delta)p = 0$$

- $\boxed{-\omega^2 = (s + 2\mathrm{i}\pi f)^2}$
- $s = 0$: Fourier domain $\omega^2 = (2\pi f)^2$,
- $f = 0$: Laplace domain $-\omega^2 = s^2$.

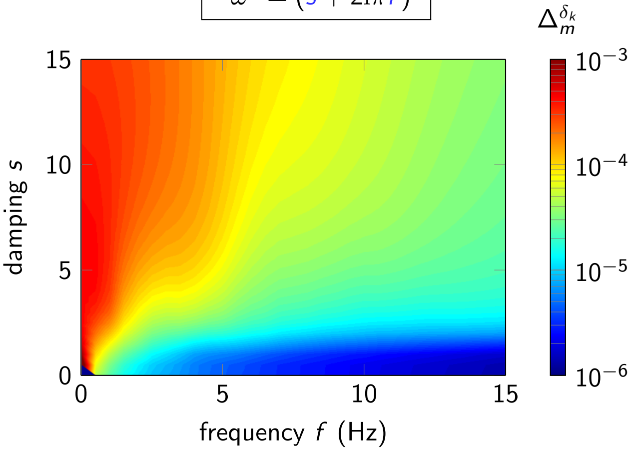
C. Shin and Y. H. Cha
Waveform inversion in the laplace domain ; Waveform inversion in the laplace fourier domain
Geophysical Journal International 2008–2009

W. Ha, S. Pyun, J. Yoo and C. Shin
Acoustic full waveform inversion of synthetic land and marine data in the laplace domain ;
Geophysical Prospecting 2010

P. V. Petrov and G. A. Newman
Three-dimensional inverse modelling of damped elastic wave propagation in the fourier domain.
Geophysical Journal International 2014

Complex frequency

$$\boxed{-\omega^2 = (s + 2i\pi f)^2}$$



Selection of complex frequency.

Stability with frequency

2/ Stability

- ensures reconstruction accuracy.

The stability indicates how minimizing the data recovers the model

$$\|m_1 - m_2\| \leq \mathcal{C} \|\mathcal{F}(m_1) - \mathcal{F}(m_2)\|.$$

The stability constant $\mathcal{C}$ depends on the frequency and the number of unknowns.

G. Alessandrini
Stable determination of conductivity by boundary measurement
Applicable Analysis 1988

E. Beretta, M. V. de Hoop, F. and O. Scherzer
Inverse boundary value problem for the Helmholtz equation: quantitative conditional Lipschitz stability estimates.
SIAM J. Math. Analysis 2016

G. Alessandrini, M.V. de Hoop, R. Gaburro and E. Sincich
Lipschitz stability for a piecewise linear Schrödinger potential from local Cauchy data
arXiv:1702.04222, 2017

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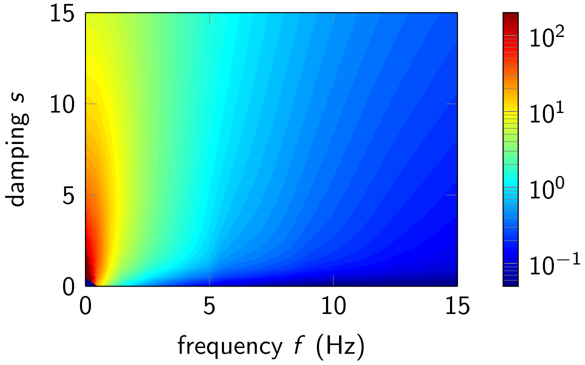
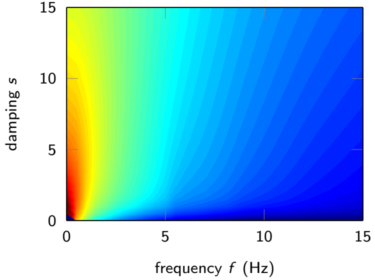
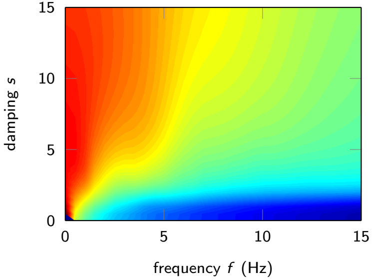


Stability & Convergence estimates

From our estimates, we see different conditions for stability and convergence.



(a) stability



(b) convergence

Quantitative estimates provide an initial, comprehensive, relation to guide the iterative procedure.

Plan

3. Numerical experiments

Multi-level algorithm: stability and convergence

0./ No initial information on the subsurface

largest radius of convergence,

1./ Start with low frequency

high stability constant.

Multi-level algorithm: stability and convergence

0./ No initial information on the subsurface

largest radius of convergence,

1./ Start with low frequency

high stability constant.

improve stability, *i.e.* resolution,

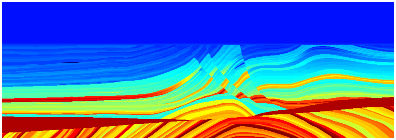
2./ Increase frequency

reduced radius of convergence.

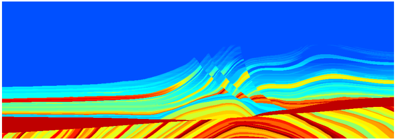
Elastic Marmousi model

Multi-parameters inversion 17×3.5km, true models.

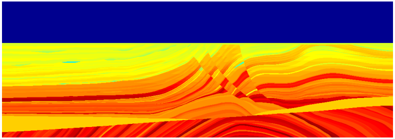
$$-\rho\omega^2 \boldsymbol{u} - \nabla\left(\lambda \nabla \cdot \boldsymbol{u}\right) - \nabla \cdot \left(\mu \left[\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T\right]\right) = 0.$$



(a) P-wave speed $\sqrt{\frac{\lambda+2\mu}{\rho}}$



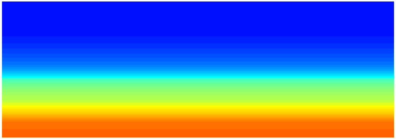
(b) S-wave speed $\sqrt{\frac{\mu}{\rho}}$



(c) Density

Elastic Marmousi model

Multi-parameters inversion 17×3.5km, starting models.



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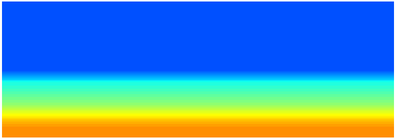

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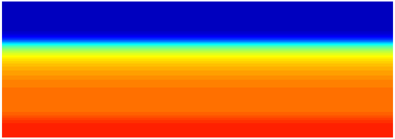
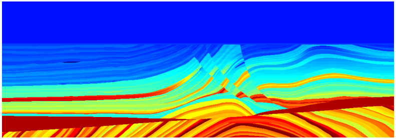


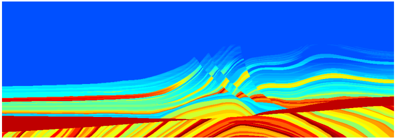
(c) Density

Elastic Marmousi reconstruction 17×3.5km

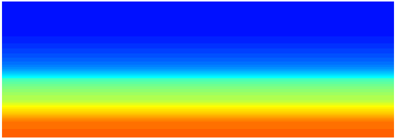
Frequency from 1 to 10Hz, unknown density, multi-level algorithm.



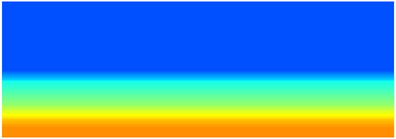
(a) True P-wave speed



(b) True S-wave speed



(c) Initial P-wave speed



(d) Initial S-wave speed

Elastic Marmousi reconstruction 17×3.5km

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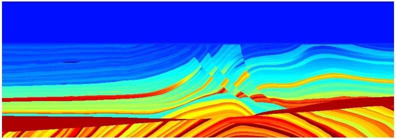

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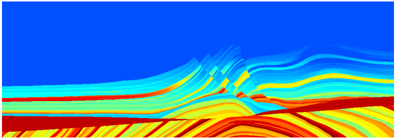
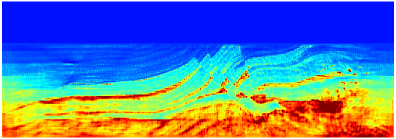
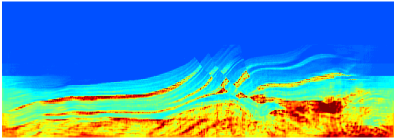


(b) True S-wave speed



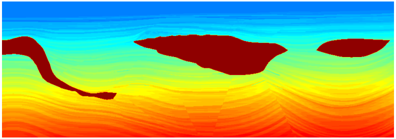
(c) 10Hz P-wave speed



(d) 10Hz S-wave speed

Elastic Pluto models

Multi-parameters inversion 31×7km, true models.



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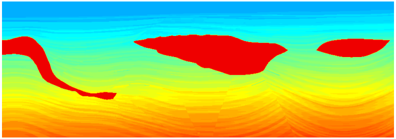
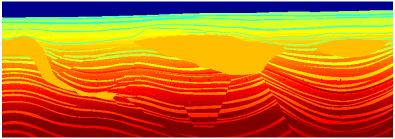


(b) S-wave speed



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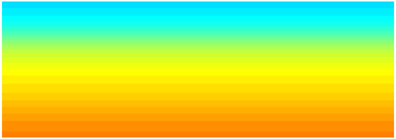

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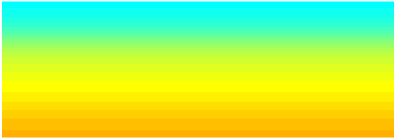
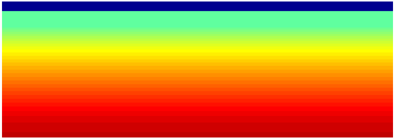


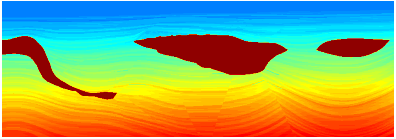
(b) S-wave speed



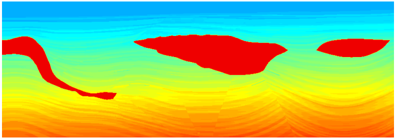
(c) Density

Elastic Pluto reconstruction 31 × 7km

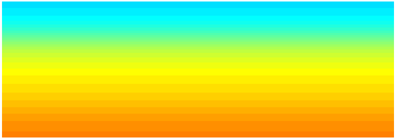
Frequency from 1 to 10Hz, unknown density, multi-level algorithm.



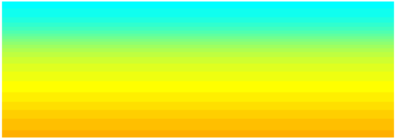
(a) True P-wave speed



(b) True S-wave speed



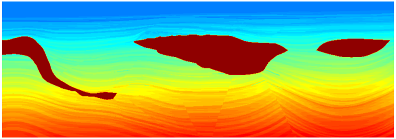
(c) Initial P-wave speed



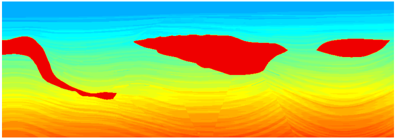
(d) Initial S-wave speed

Elastic Pluto reconstruction 31 × 7km

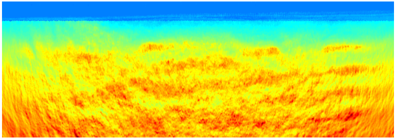
Frequency from 1 to 10Hz, unknown density, multi-level algorithm.



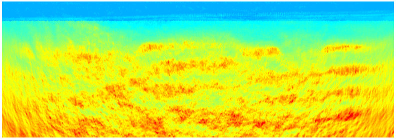
(a) True P-wave speed



(b) True S-wave speed



(c) 10Hz P-wave speed

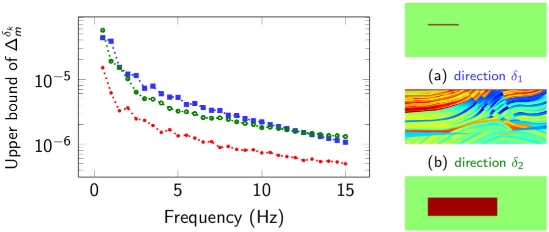


(d) 10Hz S-wave speed

Elastic Pluto reconstruction 31 × 7km

Reminder from the convergence analysis.

Estimation of the basin of attraction



(a) direction δ_1

(b) direction δ_2

(c) direction δ_3

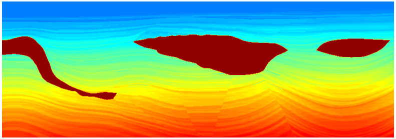
- Salt domes reduce the size of the radius of convergence.

Solution

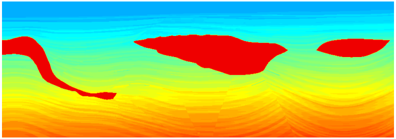
- Start with low or complex frequency.

Elastic Pluto reconstruction 31×7km

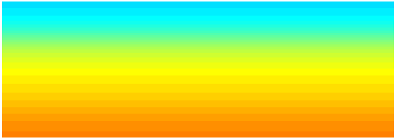
Using complex frequencies, unknown density, multi-level algorithm.



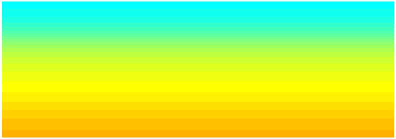
(a) True P-wave speed



(b) True S-wave speed



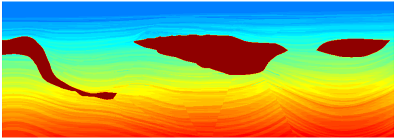
(c) Initial P-wave speed



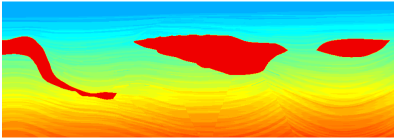
(d) Initial S-wave speed

Elastic Pluto reconstruction 31 × 7km

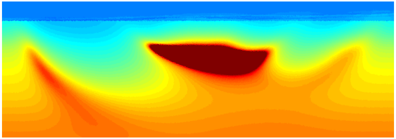
Using complex frequencies, unknown density, multi-level algorithm.



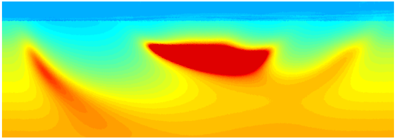
(a) True P-wave speed



(b) True S-wave speed



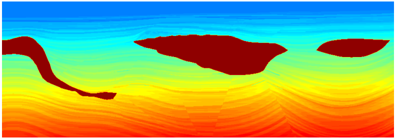
(c) (0Hz,10) P-wave speed



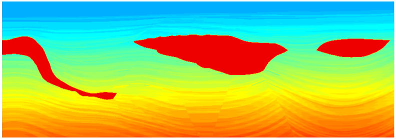
(d) (0Hz,10) S-wave speed

Elastic Pluto reconstruction 31 $\times$ 7km

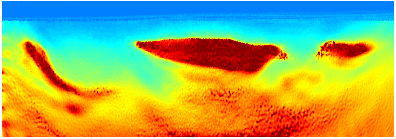
Using complex frequencies, unknown density, multi-level algorithm.



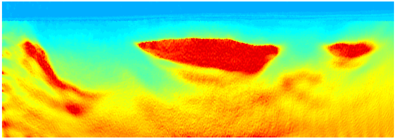
(a) True P-wave speed



(b) True S-wave speed



(c) 10Hz P-wave speed



(d) 10Hz S-wave speed

Plan

4 Conclusion

Conclusion

Comprehensive FWI

- Quantitative stability and convergence analysis:
  - depending on the frequency and geometry,
  - **depending on the methods**.
- We can also quantify the noise robustness.

Perspectives

- Analytical relation to develop between the two components,
- obtain conditional progression in frequency from stability and convergence estimates

Quantitative reconstruction method for inverse problem

- Discontinuous Galerkin discretization in HPC framework,
- acoustic, elastic, anisotropy, viscosity; 2D, 3D, dual-sensors data.

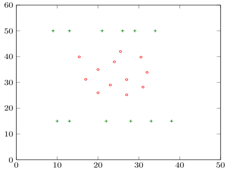
Applications and ongoing investigations:

Quantitative reconstruction method for inverse problem

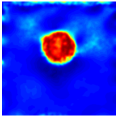
- Discontinuous Galerkin discretization in HPC framework,
- acoustic, elastic, anisotropy, viscosity; 2D, 3D, dual-sensors data.

Applications and ongoing investigations:

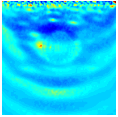
- **Inverse scattering** using obstacles positions,



- **visco-elastic** reconstruction: five unknowns,



(a) λ



(b) Q_λ

- $(\lambda,\ \mu,\ \rho,\ Q_\lambda,\ Q_\mu)$,
- unknown Q does not prevent λ recovery,
- procedure to recover attenuation parameter?

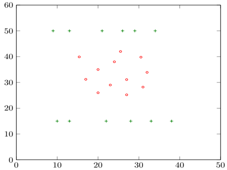
- **helioseismology**; **anisotropy**; model **parametrization**.

Quantitative reconstruction method for inverse problem

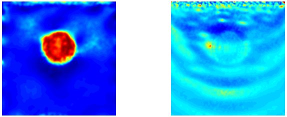
- Discontinuous Galerkin discretization in HPC framework,
- acoustic, elastic, anisotropy, viscosity; 2D, 3D, dual-sensors data.

Applications and ongoing investigations:

- **Inverse scattering** using obstacles positions,



- **visco-elastic** reconstruction: five unknowns,



(a) λ (b) Q_λ

- $(\lambda,\ \mu,\ \rho,\ Q_\lambda,\ Q_\mu)$,
- unknown Q does not prevent λ recovery,
- procedure to recover attenuation parameter?

- **helioseismology**; **anisotropy**; model **parametrization**.

THANK YOU

Appendix

Quantitative convergence for MBTT

Numerical estimate of the size $\Delta_m^{\delta_k}$ the problem components

1. the frequency ω,
2. with direction δ_k (the geometry of the unknown),
3. with the parametrization.

The estimates allow a comparison of methods, in particular we want to compare the dependency of the optimization on the low frequencies.

Quantitative convergence for MBTT

Numerical estimate of the size $\Delta_m^{\delta_k}$ the problem components

1. the frequency ω,
2. with direction δ_k (the geometry of the unknown),
3. with the parametrization.

the MBTT method decomposes the model with a smooth part (propagator) and the reflectors:

$$c^{-2} = p + r = p + D\mathcal{F}^*(p)s.$$

F. Clément and G. Chavent
Waveform inversion through MBTT formulation – 1992

F. Clément, G. Chavent and S. Gómez
Migration-based traveltime waveform inversion of 2-d simple structures: A synthetic example
Geophysics 2001

G. Chavent, K. Gadylshin and V. Tcheverda
Reflection fwi in mbtt formulation
EAGE 2015

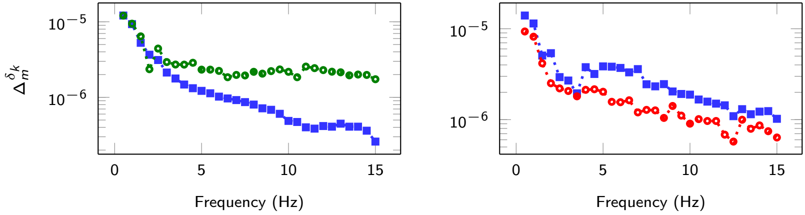
Quantitative convergence for MBTT

Numerical estimate of the size $\Delta_m^{\delta_k}$ the problem components

1. the frequency ω,
2. with direction δ_k (the geometry of the unknown),
3. with the parametrization.

Comparison of convergence with parametrization,

$$c^{-2} = p + D\mathcal{F}^*(p)s.$$



Quantitative convergence estimates with frequency

Frequency progression for iterative algorithm, $-\omega^2 = (s + 2\mathrm{i}\pi f)^2$.

Other application

- Single frequency gives high radius than range of frequencies,
- can be apply for other minimization problem,
- ...

Finite curvature indicates robustness to noise

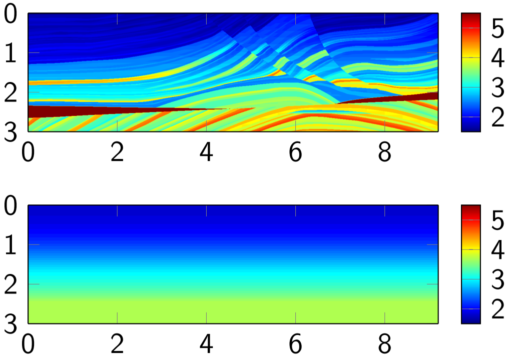
- Low and complex frequencies more affected.

Stability

- Indicates the accuracy of the reconstruction.

Acoustic Marmousi reconstruction 9.2×3km

Starting available frequency is 4Hz.



Acoustic Marmousi reconstruction 9.2×3km

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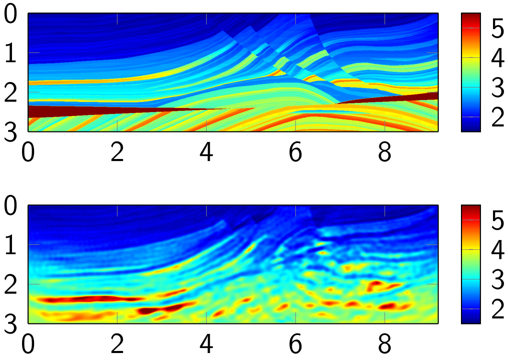
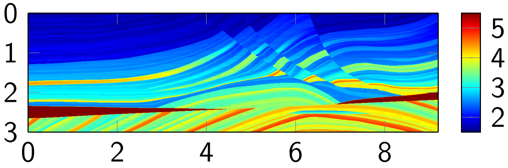
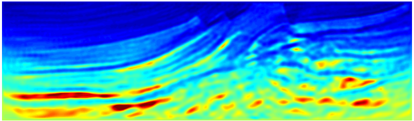


Using Fourier frequencies only: from 4 to 10Hz

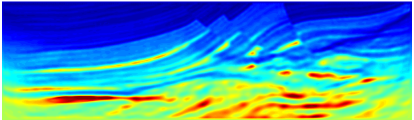
Acoustic Marmousi reconstruction 9.2×3km

Starting available frequency is 4Hz.





(a) Using Fourier frequencies only



(b) Using Complex frequencies from (4Hz, 7)